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COMMUTATIVE POLYNOMIAL SEMIGROUPS ON A SEGMENT

P.C. BAAYEN and Z. HEDRLÍN, Amsterdam, Praha

1. Introduction

A commutative semigroup of mappings of a set X is a family of mappings $X \to X$ which is a commutative semigroup under composition of functions. A commutative polynomial semigroup of mappings of a subset X of the real line R (shortly: an X-cps) is a commutative semigroup of mappings $X \to X$, all elements of which are restrictions to X of (real) polynomials on R. Such a semigroup S is called maximal if every continuous map $g : X \to X$ which commutes with all $f \in S$ itself belongs to S, and entire if it contains (restrictions to X of) polynomials of every non-negative degree.

If S_i is a semigroup of continuous maps $X_i \to X_i$ ($i = 1,2$), and if τ is a homeomorphism of X_1 onto X_2 such that $S_2 = \{\tau \circ f \circ \tau^{-1} / f \in S_1\}$, then S_1 and S_2 are called equivalent (by means of τ). In that case the transformation $f \to \tau \circ f \circ \tau^{-1}$ is an isomorphism of the abstract semigroup S_1 onto the abstract semigroup S_2.

In this note we determine, up to equivalence, all entire I-cps, where I is the closed unit segment $[0, 1]$. Moreover, we establish which of these I-cps are maximal and which not. We denote by J the segment $[-1, 1]$.

2. Commutative polynomial semigroups of mappings $R \to R$ and $J \to J$.

It follows from results of J.F. Ritt [7, 8] and of H.D.

Block and H.P. Thielman [5] that every entire R-cps is equivalent by means of a linear transformation to one of the following three semigroups of polynomials:

(i) the semigroup P, consisting of the maps $P_o, P_1, P_2, \dots$ with

$$P_n(x) = x^n ;$$

(ii) the semigroup P^*, consisting of all P_n, $n \geqslant 1$ and the map P_o^* such that

$$P_o(x) = 0 \text{ for all } x ;$$

(iii) the semigroup T of all Chebyshev polynomials $T_o, T_1, T_2, \dots$, where

$$T_n(x) = \cos (n. \text{ arc } \cos x) .$$

The first two semigroups are not maximal; e.g. consider $x^{\frac{2}{3}}$.

<u>Lemma 1</u>. There exists a unique maximal commutative semigroup $\overline{P}$ ($\overline{P^*}$) of continuous maps $J \to J$ containing $P|J$ ($P^*|J$, respectively). The semigroup $\overline{P}$ ($\overline{P^*}$) consists of the following maps: all maps $x \to |x|^{\varepsilon}$, $\varepsilon > 0$ a real number; all maps $x \to |x|^{\varepsilon} \cdot \operatorname{sign} x$, $\varepsilon > 0$ a real number; and all maps in P (in P^*, respectively).

<u>Proof</u>. It is immediately verified that $\overline{P}$ and P^* are commutative semigroups. In order to show their maximality, and the fact that they are the only maximal semigroups containing $\overline{P}$ or $\overline{P^*}$, we proceed as follows.

Let f be any continuous map $R \to R$ commuting with all maps in P or in P^*. Take any a with $0 < a < 1$ and let $f(a) = \alpha$. As $\alpha = P_2 f(\sqrt{a})$, $\alpha \geqslant 0$ if $\alpha = 0$, it follows that $f(a^r) = \alpha^r = 0$ for all rational r, because $f \circ P_n = P_n \circ f$ for all natural n. Hence $f(x) = 0$ for $x \geqslant 0$; if $x \leqslant 0$, $P_2 f(x) = f(x^2) = 0$ implies again $f(x) = 0$. Thus f is identically zero.

Assume $\alpha > 0$ and let $\varepsilon \in R$ with $a^{\varepsilon} = \alpha$. Then as f and P_n commute, $f(a^r) = a^{r\varepsilon}$ for all rational r ; hence $f(x) = x^{\varepsilon}$ for $x \geq 0$. If $x < 0$, then $P_2 f(x) = fP_2(x) = (x^2)^{\varepsilon}$, hence $f(x) = \pm|x|^{\varepsilon}$. As f is continuous, the lemma follows.

The situation is different for the semigroup T : this semigroup is maximal. In order to show this, we consider the following mappings of the unit interval I into itself, first introduced in [2]:

$$t_0(x) = 0 \text{ for all } x ;$$

and, if $n \geq 1$:

$$\begin{cases} t_n(\frac{2k}{n}) = 0 , \quad t_n(\frac{2k+1}{n}) = 1 \quad (k = 0,1,2,\dots,[\frac{n}{2}]) ; \\ t_n \mid [\frac{k}{n}, \frac{k+1}{n}] \text{ is linear } \quad (k = 0,1,2, \dots, n-1) . \end{cases}$$

These so-called multihats are easily seen to constitute a commutative semigroup M ; in fact, $t_n \circ t_m = t_{n+m}$. In [2] P.C. Baayen, W. Kuyk and M.A. Maurice proved much more: the semigroup of all t_n, $n = 0,1,2, \dots$, <u>is a maximal</u> commutative semigroup of continuous maps $I \to I$.

<u>Lemma 2</u>. The semigroup M is equivalent to the semigroup T' of all Chebyshev polynomials T_n, restricted to the segment J, by means of the homeomorphism $\tau : [0,1] \to [-1, 1]$ such that

$$\tau x = \cos \pi x .$$

<u>Proof</u>: immediate.

Hence we have shown:

<u>Lemma 3</u>. The J-cps T is maximal.

This strengthens considerably a result of G. Baxter and J.T. Joichi [3], who showed that T cannot be embedded in a 1-parameter semigroup of commuting functions.

We conclude this section with a triviality.

Lemma 4. Let Q_1, Q_2 be polynomials commuting on some non-degenerate segment. Then Q_1 and Q_2 commute everywhere on R.

3. Commutative polynomial semigroups of mappings $I \to I$

It follows from the results of section 2 that every entire I-cps is equivalent by means of a linear transformation to a semigroup $S|A$, where S is one of the R-cps T, P, P^* and A is a closed segment $[a, b]$, $a < b$, that is invariant under S.

The only non-degenerate segment mapped into itself by T is $[-1, +1]$. The only non-trivial segments mapped into themselves by P are the segments $[-a, 1]$, with $0 \leq a \leq 1$; we write $P(a)$ for the $[-a, 1]$-cps of all $P_n | [-a, 1]$, $n = 0,1,2, \ldots$. The only non-trivial segments invariant under P^* are the segments $[-a, b]$, with $0 \leq a \leq 1$, $a^2 \leq b \leq 1$, $b \neq 0$; we write $P^*(a, b)$ for the $[-a, b]$-cps of all $P_n | [-a, b]$, $n \geq 1$ together with $P_0^* | [-a, b]$.

Lemma 5. Each of the semigroups $P(a)$, $0 \leq a \leq 1$, is not maximal, and is contained in a unique maximal $[-a, 1]$-semigroup $\overline{P(a)}$. Similarly each $P^*(a, b)$ is contained in a unique maximal $[-a, b]$-semigroup $\overline{P^*(a, b)}$.

Proof. In the same way as in the proof of Lemma 1 one shows that $\overline{P(a)} = \overline{P} \| [-a, 1]$ is the unique maximal commutative semigroup of continuous maps $[-a, 1] \to [-a, 1]$ containing $P(a)$. Similarly $\overline{P^*(a, b)} = \overline{P^*} \| [-a, b]$.

Remark: If S is a semigroup of mappings of a set X into itself, and if $A \subset X$, then $S \| A$ denotes the semigroups of mappings of A into itself, consisting of all mappings $f|A$ such that $f \in S$ and $f(A) \subset A$ (cf.[6]).

Theorem 1. There are two maximal entire I-cps; they are both equivalent to T' (or to M).

Proof. Every maximal entire I-cps must be equivalent by means of a linear map to $T' = T \mid [-1, +1]$. There exist two linear maps of $[-1, +1]$ onto $I = [0, 1]$.

Lemma 6. If $0 < a, b < 1$, then $P(a)$ and $P(b)$ are equivalent by means of the homeomorphism τ,

$$\tau(x) = \operatorname{sign} x . |x|^{\varepsilon},$$

where $\varepsilon = \dfrac{\log b}{\log a}$.

Lemma 7. Let $0 \leq a_i \leq 1$, $a_i^2 \leq b_i \leq 1$, $b_i \neq 0$ $(i = 1,2)$. The semigroups $P^*(a_1, b_1)$ and $P^*(a_2, b_2)$ are equivalent if and only if there exists a real number $\varepsilon \neq 0$ such that $a_2 = a_1^{\varepsilon}$, $b_2 = b_1^{\varepsilon}$.

Proof. Suppose $P^*(a_1, b_1)$ and $P^*(a_2, b_2)$ are equivalent by means of τ. Then we have, for arbitrary $x \in [-a_1, b_1]$ and for arbitrary integers $n \geq 1$, that $P_n(x) = (\tau^{-1} \circ P_n \circ \tau)(x)$; i.e. $(\tau \circ P_n)(x) = (P_n \circ \tau)(x)$. It follows (cf. lemma 1) that either τ is of the form: $\tau(x) = |x|^{\varepsilon}$, for all $x \in [-a_1, b_1]$, where ε is some real number - as τ is a homeomorphism this is only possible if $a_1 = 0$ - or τ is of the form: $\tau(x) = |x|^{\varepsilon} . \operatorname{sign} x$. As clearly we must have: $\tau(a_1) = a_2$, $\tau(b_1) = b_2$, the assertion follows.

The next lemma is easily proved:

Lemma 8. No semigroup $P(a)$ is equivalent to a semigroup $P^*(b, c)$.

Consequently we have:

Theorem 2. There are infinitely many non-equivalent non-maximal entire I-cps. Each of them is equivalent to one of the following semigroups, which are all mutually inequivalent: $P(0)$,

$P(\frac{1}{2})$, $P(1)$; $P^*(a, 1)$, $0 \leq a \leq 1$; $P^*(a, \frac{1}{4})$, $0 \leq a \leq \frac{1}{2}$.

Theorem 3. Every entire I-cps is contained in a unique maximal commutative semigroup of continuous maps $I \to I$. Two entire I-cps are equivalent if and only if the maximal commutative semigroups in which they are contained are equivalent.

4. Remark on mappings commuting with T_n or P_n, $n \geq 2$.

It was shown by P.C. Baayen and W. Kuyk im [1] that every open map of I into itself that commutes with t_2 is itself a multihat t_n . From this it follows almost at once that every continuous map commuting with t_2 is either a t_n or is everywhere oscillating (nowhere monotone).

This result has been improved very much by G. Baxter and J.T. Joichi [4], who showed the following theorem

If a continuous map $f : I \to I$ commutes with some multihat t_n , $n \geq 2$, it is itself either a hat-function or a constant map.

Now we saw in section 2 that the semigroup M of all hats t_n is equivalent to the semigroup T' of all Chebyshev polynomials on $[-1, +1]$.

Hence we conclude:

Theorem 4. Every non-constant continuous map of $[-1, +1]$ into itself that commutes with a Chebyshev polynomial T_n with $n \geq 2$, is itself a Chebyshev polynomial.

For the maps P_n, $n \geq 2$, the situation is completely different. Consider e.g. continuous maps of $[0, 1]$ into itself which commute with P_2 on that interval.

There exist multitudes of such functions. For let $0 < a < 1$, and let f_o be any continuous function of $[a^2, a]$ into $(0, 1)$ such that $[f_o(a)]^2 = f_o(a^2)$. If we define:

$f(o) = 0$, $f(1) = 1$, $f(x) = [f_o(x^{2^{-n}})]^{2^n}$ if $x \in [a^{2^{n+1}}, a^{2^n}]$

(n integer), f will be a continuous map $I \to I$ commuting with P_2 .

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